

Outdoors Triangulation Using Rational Trigonometry

(Note: I am writing this up as an exercise mainly for my own benefit, based on some solutions that Robert Gatliff provided to me after the solutions had eluded me.)

(Second Note: Prof. Wildberger addresses this problem in his video on “Surveying.” He is solving the problem in the vertical dimension, rather than the horizontal dimension, but the math is the same. Wildberger stops at the establishment of a quadratic equation that should be solved. We have taken this one step further by applying the quadratic equation to create a “formulaic” algorithm, and understanding how to choose the appropriate solution. This may be friendlier to the person who is really looking for an algorithm (in the sense of a step-by-step process that a human can follow), but is not at all clearly superior in other respects. Additionally, we compare this to the classical formula of for the same problem, which involves SIN and therefore cannot produce exact solutions.)

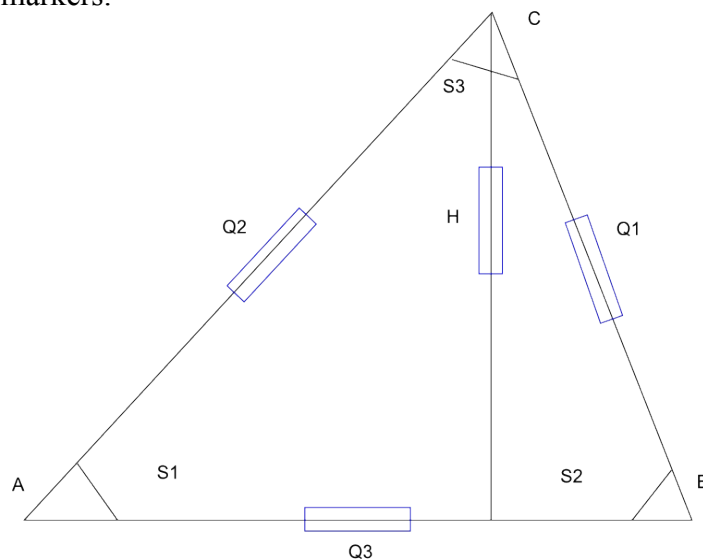
Triangulation in this instance means finding the location of a distant point by measuring its direction from two distinct known locations.

Suppose you are driving down a nice, flat, straight West Texas highway and off in the distance you see a hill rising from the plain and you wonder how far away it is.

Since here in Texas we have little green mile-markers every mile on the road, and since in West Texas at least the roads are pretty straight, you can pretty easily drive to a mile-marker, note the precise direction of the hill peak, drive to the next mile-marker and make a second measurement of the direction. (We will ignore that the peak of a hill is probably not at the same elevation as the road.)

I use the term *direction* to avoid saying *angle*. In the unlikely event you have a protractor, you can line it up against the road direction and read the angle to the hill top. (If you have a nice compass, the kind that has a little mirror and or lens and a little sighting thread, you can be more accurate, but you will measure the angle against North in both cases, not the road direction.)

Let us assume the even less likely fact that you have a *spread protractor* and can thus measure the spreads at the two mile-markers.



Using this diagram, let's assume that the road is the line AB, and that we have made the measurements S1 and S2 of the hilltop which is at C. Q3, S1, and S2 are thus knowns, and we seek to determine H. By applying the triple spread law (in particular, as expressed in exercise 7.6, as a quadratic equation that becomes):

$$s3 = s1 + s2 - 2 \cdot s1 \cdot s2 \pm 2 \cdot \sqrt{s1^2 \cdot s2^2 - s1^2 \cdot s2 - s1 \cdot s2^2 + s1 \cdot s2}$$

Robert Gatliff found a more elegant solution for the part under the root using Mathematica online:

$$s3 = s1 + s2 - 2 \cdot s1 \cdot s2 \pm 2 \cdot \sqrt{s1 \cdot (1 - s1) \cdot s2 \cdot (1 - s2)}$$

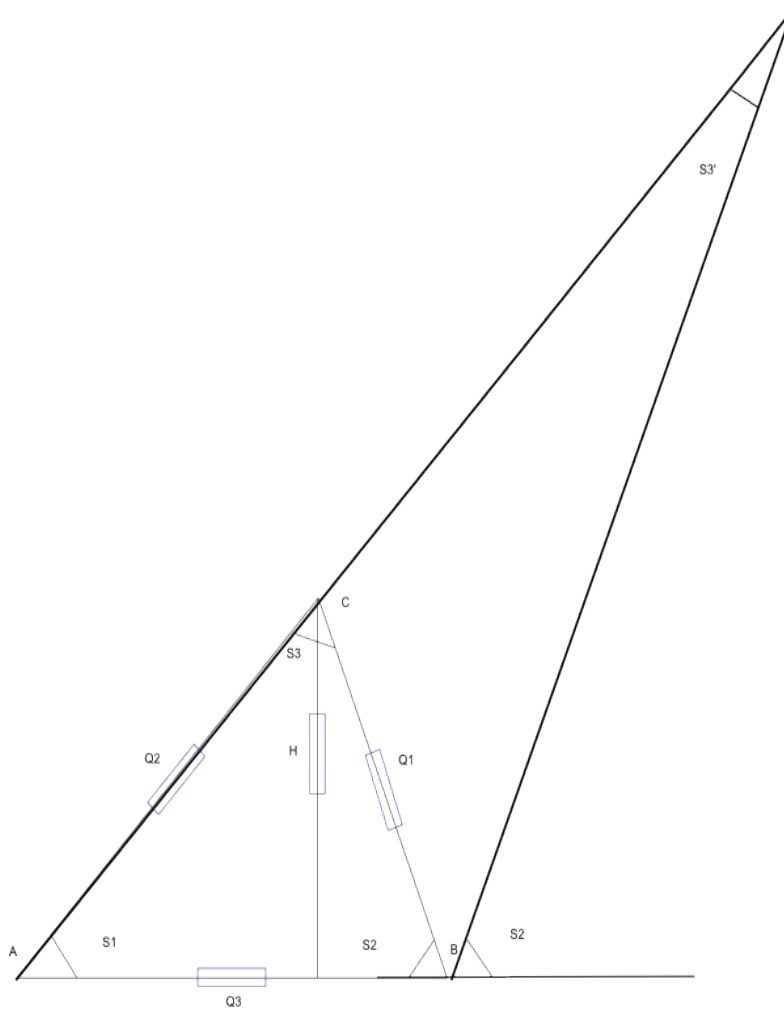
which allows us to rewrite this as:

$$s3 = s1 + s2 - 2 \cdot s1 \cdot s2 \pm 2 \cdot \sqrt{s1 \cdot (1 - s1)} \cdot \sqrt{s2 \cdot (1 - s2)}$$

Since $f(x) = \text{SQRT}(x \cdot (1 - x))$ is a simple function only of x, we could easily construct a table of it similar to a "SIN" table. This means that s3 could be calculated using table lookups.

Gatliff also pointed out that this quantity could simply be MARKED on our spread protractor! Thus, assuming we have enough room, we could mark this on our protractor for each sighting mark that we added to the protractor, and the s3 can be computed as a simple arithmetic expression.

We do have the problem that there are two solutions for s3, as illustrated in the following diagram, and captured by the (+/-) symbol before the root which shows what I call the "inside" solution and the "outside" solution (relative to the perpendiculars to the baseline through A and B).



S3' is the “outside” solution, and is narrower spread than S3, the “inside” solution. In practice, I don't think you could get these two points confused very easily. By using a quarter-circle protractor rather than a semi-circle protractor, this would be come even clearer to the user.

Which do we really intend? Clearly the expression: $2 \cdot \sqrt{s1 \cdot (1 - s1)} \cdot \sqrt{s2 \cdot (1 - s2)}$ is positive.

The expression $s1 + s2 - 2 \cdot s1 \cdot s2$ is also positive and ≤ 1.0 (see proof below.)

So the we can easily choose the solution that either makes s3 smaller, or makes s3 bigger. There will always be one solution which is between the perpendicular lines through the points A and B, and one solution outside of these. The one INSIDE will always have the larger spread (need to prove this), and thus is the (+) solution, while the one outside is the (-) solution. It seems that in general one will know which of these is intended when you take the measurement. The chance of getting is wrong is about the same as the chance of measuring the angle from the wrong direction, an equal potential error in the classical approach.

Now that we have s3 and q3, we can easily find q1 and q2 via the spread law:
 $s1/Q1 = s2/Q2 = s3/Q3$

By definition, $H = s1 \cdot Q2$

So in a single formula:

$$H = Q3 \cdot s1 \cdot s2 / (s1 + s2 - 2 \cdot s1 \cdot s2 \pm 2 \cdot \sqrt{s1 \cdot (1 - s1)} \cdot \sqrt{s2 \cdot (1 - s2)})$$

This expression is undefined only if $s1$ and $s2$ are both perpendicular to AB – which means the object is immeasurably far away, or exceeds the accuracy of our protractor (which is also true of the traditional approach.)

The distance of the object d is the square root of H away from the baseline.

By comparison, normal triangulation is given by the formula:

$$d = L \cdot \sin(a) \cdot \sin(b) / \sin(a + b)$$

Which seems simpler, and gives a unique solution. However, one has to remember to measure a and b correctly, which I think is more precisely the angle CAB and the angle BAC. Since angle defines a unique line through A and B, we don't have to worry about two solutions. Both methods in practice appear to require a calculator or a table. I would rather deal with the square root above than try to regenerate my own SIN table if I had neither, but in practice it doesn't seem to matter.

The rational method does have the advantage of producing a more understandable closed-form solution in many cases (that is, a solution involving only the square root operation and rationals, with the possibility that square root will in fact end up being completely a rational, if it happens to be a square) where as SIN forces you to always deal with approximations. It is possible that if we were solving the triangulation problem for astronomical objects (which is in principle exactly the same, thought called *parallax*), this method might be better.

It might also be computationally simpler---the only time I can think that this would be an issue is in attempting to compute the position of swarms of objects---which could in fact become important in the future.

For practical purposes for this particular problem, the traditional method seems easier than the rational trigonometry method.

Here is a little informal argument about the expression $s1 + s2 - 2 \cdot s1 \cdot s2$

Theorem: $0 \leq x \leq 1$ and $0 \leq y \leq 1$ implies that $x + y - 2 \cdot x \cdot y \geq 0$.

Proof: $x \cdot y < x$, since $y < 1$. $x \cdot y < y$, so $x \cdot y < (x+y)$. so $x + y - 2 \cdot x \cdot y > x \cdot y > 0$.

Theorem: $0 \leq x \leq 1$ and $0 \leq y \leq 1$ implies that $x + y - 2 \cdot x \cdot y \leq 1$.

$$Q = x + y - 2 \cdot x \cdot y = x + y \cdot (1 - 2 \cdot x)$$

Assume $x \geq 1/2$. Then Q is maximized when y is 0.0, but then $Q = x$, which is ≤ 1.0 .

Assume $x < \frac{1}{2}$. The Q is maximized when y is 1.0 so $Q = x + (1 - 2x) = 1 - x$, but is ≤ 1.0 .